Covariant interactions in lightcone dynamics

1993 J. Phys. A: Math. Gen. 26 5045

(http://iopscience.iop.org/0305-4470/26/19/033)

J. Phys. A: Math. Gen. 26 (1993) 5045–5055. Printed in the UK

Covariant interactions in lightcone dynamics

S N Mosley
Department of Mathematics, University of Nottingham, Nottingham NG7 2RD, UK

Received 20 November 1992, in final form 5 April 1993

Abstract. We present a classical direct-interaction theory in which the particle variables are considered over an observer's past lightcone instead of the usual constant-time hyperplanes. Because a past lightcone is Lorentz-invariant we avoid the usual problems of the simultaneity of the interaction for differently boosted observers. The theory is covariant in the sense that we show a two-particle realization of the Poincaré group with interaction. We calculate the forces due to the interaction and find tensor force equations of type $\dot{p} = F \cdot u$. For a Coulomb distance potential, the resulting force tensor can be compared directly with the Maxwell tensor as a special case.

1. Introduction

1.1. Covariant interaction theories

The special theory of relativity requires physical laws to be invariant under transformations from one inertial coordinate system to another. In the Hamiltonian formalism this implies that the Poisson bracket (PB) relations between variables are invariant. The ten fundamental generators $j^{\lambda\mu}$, p^{μ} determine how variables change under the corresponding coordinate transformation. By considering the commutation relations of the infinitessimal transformations we obtain the PB relations characteristic of the Poincaré group $\mathcal{P}$:

$$\{j^{\lambda\mu}, j^{\nu\rho}\} = \eta^{\lambda\rho} j^{\mu\nu} + \eta^{\mu\nu} j^{\lambda\rho} - \eta^{\lambda\nu} j^{\mu\rho} - \eta^{\mu\rho} j^{\lambda\nu} \tag{1.1a}$$

$$\{j^{\lambda\mu}, p^{\nu}\} = \eta^{\mu\nu} p^{\lambda} - \eta^{\lambda\nu} p^{\mu} \tag{1.1b}$$

$$\{p^{\lambda}, p^{\mu}\} = 0. \tag{1.1c}$$

Single-particle realizations of $\mathcal{P}$ are ten generators, constructed from the particle position and conjugate momentum, which satisfy (1.1). A many-particle realization of $\mathcal{P}$—with no interaction—can be obtained by simply summing the individual particle generators. If we can introduce extra 'potential' terms into the many-particle generators—these still satisfying (1.1)—then we have a *covariant interaction theory* (here we are using the term covariance in the strict sense as expounded by Dirac [1] and Foldy [2], among others).

However, covariant interaction theories in the usual spacetime coordinates face the difficulty that differently boosted observers will measure the variables on different constant-time hyperplanes. Conservation of momentum on any hyperplane (and other plausible assumptions) lead to the well-known 'no interaction' theorem of Currie *et al* [3], severely restricting any covariant interaction theory in the sense outlined above. For a lucid discussion of this theorem see, p 168–72 of Mann's book [4]. Bakamjian and Thomas [5] avoided the restrictions of the 'no-interaction' theorem by giving up the so-called 'world line conditions' (whereby differently boosted observers should calculate the same

0305-4470/93/195045+11$07.50

world lines of particles). Since this pioneering paper, there has been a considerable literature on the subject—with no agreed solution even for two particles. All attempts at constructing hyperplane interactions—with the implied simultaneity of interaction—seem hard to reconcile with electromagnetic theory, in which the force acting on one particle is calculated from the retarded variables of the other particle(s).

1.2. A review of the lightcone generators

Dirac [1] pointed out that the usual 'instant' form of Hamiltonian dynamics whereby particle variables are evaluated over the space-like hyperplanes $t = t_1, t = t_2, \cdots$ is not the only possibility. One of the the alternatives he considered was lightcone dynamics (a special case of what Dirac called the 'point' form of dynamics), whereby the particle variables are evaluated on the stack of past lightcones $T = T_1, T = T_2, \cdots$ centred on an observer's world line [6] instead of on constant-time hyperplanes. The particle information on the past lightcone $T = T_0$ is in principle available to the observer at time T_0. No observer is privileged, as by a spacetime translation any one past lightcone can be transformed into any other.

The position variable on the observer's past light-cone $T =$ constant, i.e. the position actually seen by the observer at time T, we call $\boldsymbol{y}(T)$, with $|\boldsymbol{y}| \equiv y$. The position $\boldsymbol{y}$ is the space part of the null 4-vector $y^\lambda \equiv (-y, \boldsymbol{y})$, where y^λ serves to parametrize the past lightcone with vertex at the origin. Single-free-particle realizations of $\mathcal{P}$ in lightcone coordinates have been discussed in [1,6,7]. In [7] the Hamiltonian is derived from the Lagrangian

$$L = -m\,[(1 - \frac{\boldsymbol{y}}{y} \cdot \boldsymbol{w})^2 - \boldsymbol{w}^2]^{1/2}.$$

where $\boldsymbol{w}$ is the apparent velocity $\partial \boldsymbol{y}/\partial T$. Then in terms of the conjugate momentum $\boldsymbol{\pi} \equiv \partial L/\partial \boldsymbol{w}$ the Hamiltonian is

$$H = p^0 = \frac{1}{2} y \frac{m^2 + \boldsymbol{\pi}^2}{\boldsymbol{y} \cdot \boldsymbol{\pi}} \tag{1.2a}$$

and the other generators are [6,7]

$$\boldsymbol{p} = \boldsymbol{\pi} - \frac{1}{2} \boldsymbol{y} \frac{m^2 + \boldsymbol{\pi}^2}{\boldsymbol{y} \cdot \boldsymbol{\pi}} \quad \boldsymbol{j} \equiv (j^{23}, j^{31}, j^{12}) = \boldsymbol{y} \times \boldsymbol{\pi} \quad \boldsymbol{k} \equiv (j^{10}, j^{20}, j^{30}) = y\boldsymbol{\pi}. \tag{1.2b}$$

The conjugate momentum $\boldsymbol{\pi}$ generates space translations on the past lightcone, and so is not equivalent to $\boldsymbol{p}$ which generates space translations in Minkowski 4-space. Defining the Poisson bracket as

$$\{f, g\} \equiv \left(\frac{\partial f}{\partial \boldsymbol{y}}\right) \cdot \left(\frac{\partial g}{\partial \boldsymbol{\pi}}\right) - \left(\frac{\partial f}{\partial \boldsymbol{\pi}}\right) \cdot \left(\frac{\partial g}{\partial \boldsymbol{y}}\right) \tag{1.3}$$

it may then be verified that the generators (1.2) satisfy all the PB relations (1.1), so that there is a conserved energy–momentum $p^\lambda(\boldsymbol{y}, \boldsymbol{\pi})$ covariant under Lorentz transformations. The past lightcone is Lorentz-invariant, as physically two differently boosted observers momentarily coinciding will see the same set of events, although assigning these events different positions $\boldsymbol{y}$. From (1.2) and (1.3) it follows that

$$\{j^{\lambda\mu}, y^\nu\} = \eta^{\mu\nu} y^\lambda - \eta^{\lambda\nu} y^\mu \tag{1.4}$$

as required for $y^\nu \equiv (-y, \boldsymbol{y})$ to behave as a 4-vector. Any other vector satisfying the relation equivalent to (1.4) we define to be a 4-vector, i.e. to be covariant under Lorentz transformations.

The p^μ generate a shift of the lightcone vertex through spacetime, i.e. the observer associated with the lightcone vertex moved by δa_μ will see a change in position $\delta y^\lambda = \{y^\lambda\,,\, p^\mu\}\,\delta a_\mu$. The PB relationship between y and p is

$$\{y^\lambda\,,\, p^\mu\} = -\eta^{\lambda\mu} + \frac{y^\mu p^\lambda}{y\cdot p} \tag{1.5}$$

as may be directly verified from (1.2) and (1.3). It is remarkable that the basic Poisson-bracket relationship between the lightcone position and momentum has covariant form, even though the Poisson bracket formalism is by nature noncovariant. By contrast in the usual Hamiltonian procedure the zero component of position is a parameter or zero. With the aid of (1.5) the change in y^λ due to the infinitessimal spacetime translation δa_μ is then

$$\{y^\lambda,\, p^\mu\}\,\delta a_\mu = (-\eta^{\lambda\mu} + \frac{y^\mu p^\lambda}{y\cdot p})\,\delta a_\mu = -\delta a^\lambda + p^\lambda \frac{y\cdot \delta a}{y\cdot p}\,. \tag{1.6}$$

Consider a purely spatial translation $\delta a^\mu = (0, \delta\boldsymbol{a})$, then from (1.6) the change in $\boldsymbol{y}$ is

$$\delta\boldsymbol{y} = -\delta\boldsymbol{a} - \boldsymbol{p}\left(\frac{\boldsymbol{y}\cdot\delta\boldsymbol{a}}{y\cdot p}\right) = -\delta\boldsymbol{a} + \boldsymbol{p}\left(\frac{\boldsymbol{y}\cdot\delta\boldsymbol{a}}{yp^0 + \boldsymbol{y}\cdot\boldsymbol{p}}\right) \tag{1.7}$$

so that as well as the expected $-\delta\boldsymbol{a}$ term there is an extra convection term in the direction of $\boldsymbol{p}$, due to the fact that in general a space-translated observer will not see the particle at the same point on its world line. Instead of expressing the generators (1.2) in terms of $\boldsymbol{y}, \boldsymbol{\pi}$, they may be expressed simply in terms of y, p as:

$$j^{\lambda\mu} = (y\wedge p)^{\lambda\mu} \equiv y^\lambda p^\mu - y^\mu p^\lambda \qquad p^\mu = p^\mu\,. \tag{1.8}$$

2. Particle interactions considered on an observer's past lightcone

Dirac's main interest in considering alternative forms of Hamiltonian dynamics [1] was the possibility of introducing covariant interactions as discussed in section 1.1. We will construct a two-particle realization of $\mathcal{P}$ with interaction, the dynamical variables being evaluated on past lightcones T = constant. While following in the footsteps of Dirac and Thomas [9], we believe that the specific interactions considered below are new. We procede as follows:

(1) From the single-free-particle generators $j^{\lambda\mu}$, p^μ we can construct the trivial many-particle generators $J = \sum_i j_i$, $P = \sum_i p_i$.

(2) We construct a relative-distance 4-vector $(q_i^\lambda - q_j^\lambda)$, the norm of which is a Lorentz scalar (section 3).

(3) Potential terms (functions of the distance scalar) are inserted into the translation generators while these new generators with interaction terms still satisfy (1.1). Physically the presence of interaction terms in the translation generators means that the energy–momentum of the particles alone is not conserved in the interaction region, as some of the system's

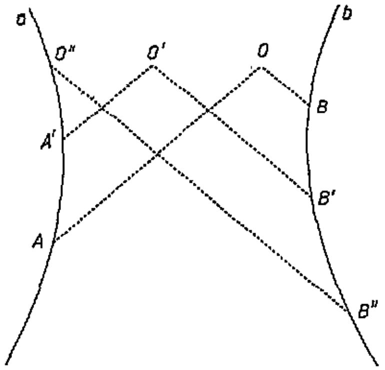

Figure 1. The particles a and b will be seen by translated observers at different points on their world lines. The total energy–momentum is the same for any inertial observer.

momentum is carried by the interaction, which is known to be the case for electromagnetic interactions (see e.g. Rosser [8] pages 298–9).

Consider the world lines of two interacting particles a and b as in figure 1. An observer at O sees the particles at positions A and B, with momenta $p(A), p(B)$. Another observer at O' sees the particles at new positions A' and B' with momenta $p(A')$ and $p(B')$. The momentum carried by the interaction is $p(\mathrm{Int})$ for O and $p(Int')$ for O'. Conservation of momentum now requires that

$$p(A) + p(B) + p(\mathrm{Int}) = p(A') + p(B') + p(\mathrm{Int}') \tag{2.1}$$

even though all three quantities are not seperately conserved. Note that the interaction momentum $p(\mathrm{Int})$ changes even for space-translated observers, as in general they see the particles at different points on their world lines and so the interparticle distance will also be different. Interactions between particles on an observer's past lightcone cannot be described by the Hamiltonian formalism in the usual Minowski coordinates, as the events A and B, etc. in figure 1 would then be at different times.

(4) The forces on a particle due to the interaction are calculated by the usual procedure, i.e. by taking the Poisson bracket (PB) of the individual particle momentum with the Hamiltonian (section 4).

(5) We compare the forces predicted for the specific case of a Coulomb potential (section 5).

We will want to compare our theory with standard electromagnetics by seeing if the predicted forces on the particles are the same. In general a straightforward comparison is impossible because standard electromagnetic theory derives the forces on one particle from information on its private past lightcone (not an observer's past lightcone). But the observer can move to coincide with one of the particles, i.e. to O'' in figure 1. Then we can directly compare the forces on particle a predicted by our theory with the standard theory. We will find that for an observer coinciding with a test particle (of negligible mass so there will be no recoil effects) then the force tensor (4.10) on the test particle is exactly as predicted by electromagnetic theory. We will thus say our theory is consistent with the usual theory for two particles, even though in general the inter-particle forces are calculated using different information. There are problems in extending the theory to more than two particles, as discussed in appendix 2. Throughout the following discussion we rely on P^{μ} as having the dual role of generating spacetime translations as well as representing conserved energy–momentum.

3. The relative position 4-vector $(q_i - q_j)$.

3.1. The projected position q_i^λ

As the position vector on the past lightcone $\boldsymbol{y}$ is part of a null 4-vector, it is unsuitable for constructing a distance between two particles. So we will construct the following 'projected position' q_i^λ of the particle i, requiring that

(1) it is a 4-vector, so that its norm will be a Lorentz-invariant distance from the origin, and

(2) in the non-relativistic limit the space components of this 4-vector reduce to $\boldsymbol{y}_i$ and the time component becomes zero.

If (1) and (2) are satisfied then in the non-relativistic limit the norm of q_i^λ is just the usual distance. The q_i^λ satisfying requirements (1) and (2) is constructed according to the following prescription (see figure 2):

> project from the particle's (retarded) position y_i^λ along the particle's (retarded) momentum p_i^λ until orthogonal to the system's momentum P^λ (3.1)

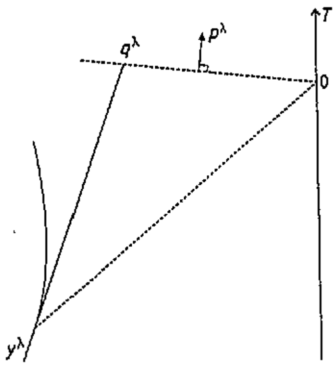

Figure 2. At time $T = 0$ a particle is seen at position $y^\lambda \equiv (-y, \boldsymbol{y})$. The 4-vector q^λ is where the tangent vector p^λ at y^λ meets the hyperplane orthogonal to the system momentum P^μ.

It follows that

$$q_i^\lambda \equiv \frac{(y_i \wedge p_i)^{\lambda\mu} \cdot P_\mu}{p_i \cdot P} = y_i^\lambda - p_i^\lambda \frac{y_i \cdot P}{p_i \cdot P} \tag{3.2}$$

where $P^\lambda \equiv \sum_i p_i^\lambda$ is the system's momentum. Note that y_i, p_i and P are all evaluated at retarded time T. The projected position $q_i^\lambda(T)$ is a 4-vector (Lorentz-covariant) by construction, being a function of the 4-vectors $y_i(T)$, $p_i(T)$ and $P(T)$, and represents the particle's position on the system's hyperplane according to the latest information available to the observer at time T, i.e. the position it would be in if its momentum remained constant. How q_i^λ transforms under translations is considered in appendix 1. The orthogonality property is easily verified:

$$q_i^\lambda P_\lambda \equiv \left(y_i^\lambda - p_i^\lambda \frac{y_i \cdot P}{p_i \cdot P}\right) P_\lambda = y_i \cdot P - y_i \cdot P \left(\frac{p_i \cdot P}{p_i \cdot P}\right) = 0 . \tag{3.3}$$

In the non-relativistic limit when $|\boldsymbol{P}| \ll P^0$, $|\boldsymbol{p}_i| \ll p_i^0$, then from (3.2) we have

$$q_i^\lambda \simeq y_i^\lambda - p_i^\lambda \left(\frac{y_i^0}{p_i^0}\right) \Rightarrow q_i^0 \simeq 0 \quad \boldsymbol{q}_i \simeq \boldsymbol{y}_i \tag{3.4}$$

so that the q satisfy condition (2) above.

3.2. Poisson bracket identities involving the q_i

In particular we derive (3.9) which enables us to introduce potential terms in the next section. In the many-particle case, the definition of the Poisson bracket becomes

$$\{f, g\} \equiv \sum_i \left(\frac{\partial f}{\partial \boldsymbol{y}_i}\right) \cdot \left(\frac{\partial g}{\partial \pi_i}\right) - \left(\frac{\partial f}{\partial \pi_i}\right) \cdot \left(\frac{\partial g}{\partial \boldsymbol{y}_i}\right). \tag{3.5}$$

We calculate the PB $\{q_i^\lambda, p_i^\mu\}$ using (1.5):

$$\begin{aligned}
\{q_i^\lambda, p_i^\mu\} &\equiv \{y_i^\lambda - p_i^\lambda \frac{y_i \cdot P}{p_i \cdot P}, p_i^\mu\} = \{y_i^\lambda, p_i^\mu\} - \frac{p_i^\lambda}{p_i \cdot P}\{y_i^\nu P_\nu, p_i^\mu\} \\
&= -\eta^{\lambda\mu} + \frac{y_i^\mu p_i^\lambda}{y_i \cdot p_i} - \frac{p_i^\lambda P_\nu}{p_i \cdot P}(-\eta^{\nu\mu} + \frac{y_i^\mu p_i^\nu}{y_i \cdot p_i}) \\
&= -\eta^{\lambda\mu} + \frac{y_i^\mu p_i^\lambda}{y_i \cdot p_i} + \frac{p_i^\lambda P^\mu}{p_i \cdot P} - \frac{p_i^\lambda y_i^\mu (p_i \cdot P)}{(p_i \cdot P)(y_i \cdot p_i)} \\
&= -\eta^{\lambda\mu} + \frac{p_i^\lambda}{p_i \cdot P} P^\mu
\end{aligned} \tag{3.6}$$

Note that since only q_i^λ contains y_i components, we have $\{q_i^\lambda, p_j^\mu\} = 0$ for $i \neq j$. Hence, we can obtain from (3.6)

$$\{q_i^\lambda, P^\mu\} = \{q_i^\lambda, p_i^\mu\} = -\eta^{\lambda\mu} + \frac{p_i^\lambda}{p_i \cdot P} P^\mu . \tag{3.7}$$

From (3.7) follows the PB between the relative position $q_i - q_j$ and the system momentum P:

$$\{q_i^\lambda - q_j^\lambda, P^\mu\} = \left(\frac{p_i^\lambda}{p_i \cdot P} - \frac{p_j^\lambda}{p_j \cdot P}\right) P^\mu \tag{3.8}$$

so that only in the system rest frame is $(q_i^\lambda - q_j^\lambda)$ invariant under space translations generated by $\boldsymbol{P}$. However we can derive from (3.8)

$$\left\{q_i^\lambda - q_j^\lambda, \frac{P^\mu}{|P|}\right\} = 0. \tag{3.9}$$

This important relationship enables us to introduce potential terms $V(q_i - q_j)$ in the next section.

Proof.

$$\{q_i^\lambda - q_j^\lambda, |P|\} = \{q_i^\lambda - q_j^\lambda, P^\mu\} \frac{P_\mu}{|P|} \tag{3.10}$$

and so

$$\{q_i^\lambda - q_j^\lambda, \frac{P^\mu}{|P|}\} = \{q_i^\lambda - q_j^\lambda, P^\mu\} \frac{1}{|P|} - \{q_i^\lambda - q_j^\lambda, |P|\} \frac{P^\mu}{|P|^2} = 0 . \tag{3.11}$$

□

From (3.7) we can calculate the rate of change of the free particle q_i position vector, where we calculate the rate of change of any variable a according to the usual prescription $\frac{da}{dT} = \frac{\partial a}{\partial T} + \{a\,,\ P^0\}$, recalling that T labels the observer's stack of lightcones:

$$\dot{q}_i^\lambda = \{q_i^\lambda\,,\ P^0\} = -\eta^{\lambda 0} + \left(\frac{p_i^\lambda}{p_i \cdot P}\right) P^0 \equiv -\eta^{\lambda 0} + u_i^\lambda \tag{3.12}$$

where we have introduced the frame-dependant velocities

$$u_i^\lambda \equiv \frac{p_i^\lambda P^0}{p_i \cdot P} \tag{3.13}$$

which reduce to the usual velocity $(1, \boldsymbol{p}_i/p_i^0)$ in the system's rest frame. We note in passing that $(q_i \wedge p_i) = (y_i \wedge p_i)$, which enables the Lorentz generators to be written as

$$\mathrm{J}^{\lambda\mu} \equiv \sum_i (y_i \wedge p_i)^{\lambda\mu} = \sum_i (q_i \wedge p_i)^{\lambda\mu}. \tag{3.14}$$

The q_i^λ of (3.3) was considered by Thomas [9] in the form $\frac{j_i^{\lambda\mu} P_\mu}{p_i \cdot P}$ which is equivalent to our definition when $j_i^{\lambda\mu}$ is the zero-spin Lorentz generator (2.5).

The analysis above leaves open the question of how the system is defined—an essential point, in view of the fact that q_i^λ depend on the momenta of all particles in the system, however far apart. In the next section, we first consider an isolated system of two particles, so that $P = p_i + p_j$ and $q_i = q_i(y_i, p_i, P) = q_i(y_i, p_i, p_j)$.

4. The two-body theory with interaction

4.1. Introduction of potential terms

The free-particle system generators for two particles i and j are simply the sum of the individual particle generators, i.e. $J^{\lambda\mu} = j_i^{\lambda\mu} + j_j^{\lambda\mu}$ and $P^\lambda = p_i^\lambda + p_j^\lambda$, so that the Poincaré group algebra (1.1) is trivially satisfied. We now use (3.9) to introduce into the 4-momentum generators P^μ potential terms $V(\bar{q}^\lambda)$, where $\bar{q}^\lambda$ is the relative position vector $(q_i^\lambda - q_j^\lambda)$ discussed in the last section. Note that as $\bar{q}$ is orthogonal to the time-like vector P then $\bar{q}$ is space-like or null, so we can define the 'distance' $\lfloor \bar{q} \rfloor \equiv (-\bar{q} \cdot \bar{q})^{1/2}$ ensuring a positive-definite quantity under the square root. The scalar $\lfloor \bar{q} \rfloor$ is manifestly Lorentz-invariant, i.e. $\{J^{\lambda\mu}\,,\ \lfloor \bar{q} \rfloor\} = 0$, but is not invariant under translations (except in the system rest frame—see 3.8). From (3.4) we see that in the non-relativistic limit $\lfloor \bar{q} \rfloor$ becomes simply the usual $|\boldsymbol{y}_i - \boldsymbol{y}_j|$. We now define the 'interaction' generators

$$P^\mu_{\text{int}} \equiv \frac{P^\mu}{|P|}(|P| + V(\lfloor \bar{q} \rfloor)) \ = \ P^\mu + \frac{P^\mu}{|P|} V(\lfloor \bar{q} \rfloor) \tag{4.1}$$

then due to (3.9) the identity $\{P^\lambda_{\text{int}}\,,\ P^\mu_{\text{int}}\} = 0$ still holds. We now have a new set of two-particle generators—with interaction—while still obeying the PB relations of the Poincaré group algebra, i.e.

$$\{J^{\lambda\mu}\,,\ J^{\nu\rho}\} = \eta^{\lambda\rho} J^{\mu\nu} + \eta^{\mu\nu} J^{\lambda\rho} - \eta^{\lambda\nu} J^{\mu\rho} - \eta^{\mu\rho} J^{\lambda\nu} \tag{4.2}$$

$$\{J^{\lambda\mu}\,,\ P^\nu_{\text{int}}\} = \eta^{\mu\nu} P^\lambda_{\text{int}} - \eta^{\lambda\nu} P^\mu_{\text{int}} \tag{4.3}$$

$$\{P^\lambda_{\text{int}}\,,\ P^\mu_{\text{int}}\} = 0 \tag{4.4}$$

with the P^{μ}_{int} defined as in (4.1). The Hamiltonian $\mathcal{H}$ is now

$$\mathcal{H} = P^0_{\text{int}} = \frac{P^0}{|P|}\left(|P| + V(\lfloor\bar{q}\rfloor)\right) \equiv \gamma\ \left(|P| + V(\lfloor\bar{q}\rfloor)\right) \tag{4.5}$$

where $\gamma = \frac{P^0}{|P|}$ is the usual 'mass increase' factor due to the centre of mass motion. The rest mass is now $(|P| + V)$—V will, of course, be negative for bound states—and the fact that the rest mass in (4.5) is multiplied by γ implies that indeed the two particles are behaving as one system.

4.2. *The equation of motion with the potential term V*

We calculate the force on the particles resulting from the distance-type potential term $V(\lfloor\bar{q}\rfloor)$. We derive the change in momentum of the ith particle using the interaction Hamiltonian (4.5):

$$\dot{p}^{\mu}_i = -\{P^0_{\text{int}}\,,\, p^{\mu}_i\} = -\gamma\,\{V(\lfloor\bar{q}\rfloor)\,,\, p^{\mu}_i\} = -\gamma\, V'(\lfloor\bar{q}\rfloor)\,\{\lfloor\bar{q}\rfloor\,,\, p^{\mu}_i\}. \tag{4.6}$$

To derive $\{\lfloor\bar{q}\rfloor\,,\, p^{\mu}_i\}$ we use (3.6):

$$\begin{aligned}\left\{\lfloor\bar{q}\rfloor\,,\, p^{\mu}_i\right\} &\equiv \left\{(-\bar{q}_{\lambda}\bar{q}^{\lambda})^{1/2}\,,\, p^{\mu}_i\right\} = -\frac{\bar{q}_{\lambda}}{\lfloor\bar{q}\rfloor}\left\{q^{\lambda}_i\,,\, p^{\mu}_i\right\} = -\frac{\bar{q}_{\lambda}}{\lfloor\bar{q}\rfloor}\left(-\eta^{\lambda\mu} + \frac{p^{\lambda}_i}{p_i\cdot P}P^{\mu}\right)\\ &= \frac{1}{\lfloor\bar{q}\rfloor}\left(\bar{q}^{\mu} - \frac{\bar{q}\cdot p_i}{p_i\cdot P}\,P^{\mu}\right) = \frac{1}{\lfloor\bar{q}\rfloor}\,\frac{\bar{q}^{\mu}\,(p_i\cdot P) - (\bar{q}\cdot p_i)\,P^{\mu}}{p_i\cdot P}\\ &= -\left(\frac{\bar{q}}{\lfloor\bar{q}\rfloor}\wedge\frac{P}{(p_i\cdot P)}\right)^{\lambda\mu}\cdot p_{i\lambda}.\end{aligned} \tag{4.7}$$

Inserting the result (4.7) into (4.6) yields

$$\begin{aligned}\dot{p}^{\mu}_i &= \gamma\, V'(\lfloor\bar{q}\rfloor)\left(\frac{\bar{q}}{\lfloor\bar{q}\rfloor}\wedge\frac{P}{(p_i\cdot P)}\right)^{\lambda\mu}\cdot p_{i\lambda} = V'(\lfloor\bar{q}\rfloor)\left(\frac{\bar{q}}{\lfloor\bar{q}\rfloor}\wedge\frac{P}{|P|}\right)^{\lambda\mu}\cdot\frac{\gamma\,|P|\,p_{i\lambda}}{(p_i\cdot P)}\\ &= V'(\lfloor\bar{q}\rfloor)\left(\frac{\bar{q}}{\lfloor\bar{q}\rfloor}\wedge\frac{P}{|P|}\right)^{\lambda\mu}\cdot u_{i\lambda} \equiv F^{\lambda\mu}\cdot u_{i\lambda}\end{aligned} \tag{4.8}$$

recalling the u^{λ} 'velocity' vectors defined by (3.13). We interpret (4.8) as implying a force tensor

$$F^{\lambda\mu} \equiv V'(\lfloor\bar{q}\rfloor)\left(\frac{\bar{q}}{\lfloor\bar{q}\rfloor}\wedge\frac{P}{|P|}\right)^{\lambda\mu}. \tag{4.9}$$

For the case of a Coulomb potential $V = e^2/\lfloor\bar{q}\rfloor$, (4.9) becomes

$$F^{\lambda\mu} \equiv -\frac{e^2}{\lfloor\bar{q}\rfloor^2}\left(\frac{\bar{q}}{\lfloor\bar{q}\rfloor}\wedge\frac{P}{|P|}\right)^{\lambda\mu}. \tag{4.10}$$

In the next section we discuss whether the $F^{\lambda\mu}$ above correponds to the usual Maxwell tensor produced by a charged particle.

5. The force tensor resulting from a Coulomb potential $V = e^2/\lfloor\bar{q}\rfloor$

As discussed at the end of section 2 the usual electromagnetic theory of interacting particles is a historical theory in the sense that the field at one particle i is calculated from the other particle variables at previous times on i's individual past lightcone. To calculate forces on both particles, information from two past lightcones is needed etc. By contrast, the theory above deduces all forces from the information on one (the observer's) past lightcone only. We can easily compare the two theories only when they are using the same information to predict a force, which only happens when the observer coincides with one of the particles; then the force on that particle is predicted by both theories using the same information. A complete theory should also take into account that interacting particles undergo acceleration, and accelerating particles always radiate according to classical theory. However, if the particle j is an ultra-massive charge, then effectively its momentum is constant and we can neglect the acceleration part of its electromagnetic field. We will see that the tensor field $F^{\lambda\mu}$ produced by an ultra-massive charge on a test charge i at the origin is *exactly* the Maxwell field as predicted by the standard electromagnetic theory.

5.1. Derivation of the the force acting on a test particle i at the origin due to particle j.

Note that

$$p_j^0 \gg p_i^0 \quad \Rightarrow \quad P \simeq p_j. \tag{5.1}$$

As particle i is at the origin,

$$q_i^\lambda = 0 \quad \Rightarrow \quad \bar{q}^\lambda = -q_j^\lambda \tag{5.2}$$

and using (5.2) with the definition (3.2) then

$$\begin{aligned}
\lfloor\bar{q}\rfloor \Rightarrow \lfloor q_j\rfloor &\equiv \left[-\left(y_j - p_j\frac{y_j\cdot p_j}{p_j^2}\right)\cdot\left(y_j - p_j\frac{y_j\cdot p_j}{p_j^2}\right)\right]^{1/2} \\
&= \left[-\left(-2\frac{(y_j\cdot p_j)^2}{p_j^2} + \frac{(y_j\cdot p_j)^2}{p_j^2}\right)\right]^{1/2} \\
&= -\frac{y_j\cdot p_j}{|p_j|} = y_j\left(\frac{p_j^0 + p_j^{\|}}{|p_j|}\right)
\end{aligned}\tag{5.3}$$

recalling that y_j is the radial distance of the particle j on the observer's past lightcone, and where $p_j^{\|} \equiv (\frac{\boldsymbol{y}_j}{y_j}\cdot\boldsymbol{p}_j)$ is the radial momentum. Now substituting (5.1), (5.2), (5.3) into (4.10), we have

$$F^{\lambda\mu} = -\frac{e^2}{\lfloor q_j\rfloor^3}\left(q_j \wedge \frac{p_j}{|p_j|}\right)^{\lambda\mu} = \frac{e^2}{(y_j\cdot\frac{p_j}{|p_j|})^3}\left(y_j \wedge \frac{p_j}{|p_j|}\right)^{\lambda\mu} \tag{5.4}$$

which agrees with the usual 'velocity' field part of the Maxwell tensor produced by particle j according to the usual theory—see for example Jackson's 2nd edition [10] p 567. However, when comparing our theory with the usual electrodynamics, it must be remembered that the force on a particle is calculated from (4.8) with respect to its frame-dependant velocity $u^\lambda \equiv p_i^\lambda P^0/(p_i\cdot P)$, which depends on the centre of mass momentum P and only reduces to the usual velocity $(1, \boldsymbol{p}_i/p_i^0)$ in the system rest frame.

If both particles have mass, then the force tensor will not have the simple form (5.4). This greater complication is reflected in the usual theory, since then both particles undergo acceleration, which means that there will be extra 'acceleration' components in the electromagnetic field produced by particle j.

6. Conclusion

We have shown that in the two-particle case the potential $V = e^2/\lfloor\bar{q}\rfloor$ introduced into the Poincaré generators effectively produces a tensor force field on the other particle. This force tensor on a test (massless) particle coinciding with the observer is exactly the Maxwell tensor predicted by the usual theory. We now resort to the covariance of our theory, and state that if the force law (4.8) with (4.10) is correct for one particular observer, then it must hold for any observer regardless of its location. Note that the Maxwell theory implies that the force on a particular particle depends on the other particle variables on that particle's individual past lightcone, not any observer's past lightcone as in our theory. So in general, the two theories calculate the force on a particle from different information.

We need to point out that there are problems in extending the interaction theory of this paper to more than two particles, as it does not satisfy the cluster decomposition property (discussed in appendix 2). However it is also known that the usual theory has problems in predicting the simplest two-particle interactions, for example the case of two particles of equal mass approaching head-on. The computations are difficult, as the differential equations involve both present and retarded times, and must be solved numerically. This has been done [11], and surprisingly the particle energies were found to increase during interaction. This result was initially attributed to not including the radiation reaction terms [11], but when these were later included energy conservation was still violated and there were run-away solutions [12]. (For a detailed recent discussion of these and other problems see pp 144–5 and pp 196–203 of [13] and references therein). In the direct interaction theory above, the energy–momentum conservation at large separations is trivially satisfied. Unfortunately, testing any interaction theory by direct experimental observation of particle trajectories is impracticable.

We expect the interaction theory presented here to be particularly applicable to bound states, noting that $V(\lfloor\bar{q}\rfloor)$ is unspecified. Quantization [7] may be formally carried out by repacing all PBs with commutators divided by i; we leave this for future consideration.

Appendix 1. How the position q_i^λ of a free particle transforms under translations

Consider a spacetime translation a_μ; then $q^\lambda \to \exp(\widetilde{a_\alpha P^\alpha})\, q^\lambda$ where we use the notation of Sudarshan and Mukunda [14]: $\exp(\tilde{A}), q^\lambda \equiv q^\lambda + \{A, q^\lambda\} + \frac{1}{2}\{A\,, \{A\,, q^\lambda\}\} + \cdots$. Using (3.7) we obtain

$$q_i^\lambda \to q_i^\lambda + a^\lambda - p_i^\lambda\left(\frac{a \cdot P}{p_i \cdot P}\right). \tag{A1.1}$$

In addition to the expected displacement a^λ, there is an another projection term proportional to the particle momentum, i.e. q_i transforms to a point on a straight line in the direction of p_i. From this we can infer that the world line q_i^λ of a particle of constant momentum is preserved under translations. This interesting fact is mentioned in the work by Bacry on a similar position generator—see footnote 13 in [15].

Appendix 2. Extension to three or more particles

We can set up N particle-generator as follows. For the free particle case, we find from section 3

$$J^{\lambda\mu} = \sum_{i=1}^{N}(y_i \wedge p_i)^{\lambda\mu} = \sum_{i=1}^{N}(q_i \wedge p_i)^{\lambda\mu} \qquad P^\mu = \sum_{i=1}^{N} p_i^\mu \tag{A2.1}$$

where the q_i 4-vectors are defined by (3.3) as functions of y_i, p_i and P. To introduce the pair interactions we label the interparticle distance 4-vectors by $\bar{q}_{ij} \equiv q_i - q_j$ etc. The relationship (3.9), which is $\{\bar{q}^{\lambda}_{ij}, P^{\mu}/|P|\} = 0$, holds for any i, j. This means that we can introduce the following set of generators (cf (4.1)) with

$$P^{\mu}_{\text{int}} = \frac{P^{\mu}}{|P|}\left(|P| + V(\bar{q}_{ij}, \bar{q}_{jk}, \cdots)\right). \tag{A2.2}$$

If V is a scalar, the PB relations of the Poincaré group (4.2)-(4.4) still hold. Specifically, for electromagnetic-type interactions we have

$$P^{\mu}_{\text{int}} = \frac{P^{\mu}}{|P|}\left(|P| + \sum_{ij} \frac{e_i e_j}{\lfloor \bar{q}_{ij} \rfloor}\right) \tag{A2.3}$$

where e_i, e_j are the particle charges.

To compare the many-body theory above with the theory of electromagnetic interactions, we note the following situation. Consider just three particles with particles i, j close together and k far away. Then particle k has no direct influence on the other two because $1/\lfloor \bar{q}_{ik} \rfloor$, $1/\lfloor \bar{q}_{jk} \rfloor \simeq 0$. But the interaction distance $\lfloor \bar{q}_{ij} \rfloor$—through which the forces of i, j acting on each other are calculated—does depend on the system's momentum which includes the momentum of particle k. In this sense the interaction is not separable however far away the third particle is.

References

[1] Dirac P A M 1949 *Rev. Mod. Phys.* **21** 392
[2] Foldy L L 1961 *Phys. Rev.* **122** 275
[3] Currie D G, Jordan T F and Sudarshan E C G 1963 *Rev. Mod. Phys.* **35** 350
[4] Mann R A 1974 *The classical dynamics of particles: Galilean and Lorentz relativity* (New York: Academic)
[5] Bakamjian B and Thomas L H 1953 *Phys. Rev.* **92** 1300
[6] Derrick G H 1987 *J. Math. Phys.* **28** 64
[7] Mosley S N and Farina J E G 1992 *J. Phys.* A **25** 4673
[8] Rosser W G V 1964 *An Introduction to the Theory of Relativity* (London: Butterworths)
[9] Thomas L H 1952 *Phys. Rev.* **85** 868
[10] Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley)
[11] Huschilt J *et al* 1973 *Phys. Rev.* D **7** 2844
[12] Kasher J C 1976 *Phys. Rev.* D **14** 939
[13] Parrott S 1987 *Relativistic Dynamics and Differential Geometry* (New York: Springer)
[14] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley) p 55
[15] Bacry H 1964 *J. Math. Phys.* **5** 109